PMM U.S.S.R.,Vol.44,pp.629-637 0021-8928/81/5 0629 $7.50/0

UDC 539.3:534.1

INFLUENCE OF INITIAL IMPERFECTIONS ON THE BUCKLING OF ELASTIC SHELLS UNDER MULTIPLE CRITICAL LOADS*

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The buckling and post-critical behavior of elastic conservative, shallow shells with very small initial imperfections in the middle-surface shape are investigated for several coincident critical loads. In this case the buckling mode of the shell in the initial post-critical stage is a linear combination of many eigenmodes and a computation of the critical loads is related to the need to solve systems of nonlinear algebraic equations /1,2/. The analysis is on the basis of the Mushtari—Donnell—Vlasov equations /3/ by the Liapunov-Schmidt operator method /4—9/. In the case of shells of arbitrary shape, asymptotic representations are constructed of new equilibria in the initial post-critical stage, a system of bifurcation equations and formulas to determine its coefficients are obtained, and equations of the critical load surfaces are also derived as functions of the shell imperfection parameters.

A complete solution of the problem is given for the non-axisymmetric buckling of the axisymmetric equilibrium of shells of revolution. Computational formulas are written down for the coefficients of the system of bifurcation equations and an algorithm is constructed to determine all its solutions. It is shown that taking account of the connectedness of the eigenmodes yields a substantial reduction in the upper critical pressure. Results of computations are presented for spherical and conical shells in two eigenmodes. According to the computations and experiments, the divergence of the theoretical values of the upper critical loads and the actual snap-through loads of a broad class of elastic shells is related mainly to small initial deviations of their shape from the assumed geometric surface /10—12/.Koiter was the first to investigate the buckling of imperfect shells, and his researches were continued by a number of authors using variational principles (see the surveys /1,2,13—16/, almost all the papers cited in these surveys are limited to a study of buckling in one eigenmode).

1. On the formulation of the problem. Operator form of the equations for the perturbations. The system of nonlinear equilibrium equations of elastic shells with initial imperfections in the middle surface shape (the Mushtari—Donnell—Vlasov variant /3/, p.101) can be represented in the form

$$\varepsilon^2\Delta^2 w - [w-z, F] + \xi[\zeta, F] = q, \quad \varepsilon^2\Delta^2 F + {}^1/_2[w, w] - [z, w] - \xi[\zeta, w] = 0 \tag{1.1}$$

$$\Delta^2 = \Delta\Delta, \quad \Delta w = l_1 w + l_2 w, \quad [w, F] = l_1 w l_2 F + l_2 w l_1 F - 2 l_3 w l_3 F, \quad l_1 w = \frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}\right) + \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta}$$

$$l_2 w = \frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) + \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha}, \quad l_3 w = -\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\partial\beta} + \frac{1}{BA^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} + \frac{1}{AB^2}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta}$$

We shall examine these equations together with each of the boundary conditions on the contour Γ (/3/)

$$1)\ w = w_{\rho\rho} - \nu\varkappa w_\rho = F = F_\rho = 0, \quad \rho = 0, \quad 2)\ w = w_\rho = F = F_\rho = 0, \quad 3)\ w = w_\rho = 0 \tag{1.2}$$

$$\Gamma_2 F = \Gamma_3 F = 0, \quad \Gamma_2 F = F_{\rho\rho} - \nu F_{ss} + \varkappa\nu F_\rho, \quad \Gamma_3 F = F_{\rho\rho\rho} + (2+\nu) F_{\rho ss} + 3\varkappa F_{ss} + (2+\nu)\varkappa_s F_s - \varkappa^2(1-\nu) F_\rho$$

All the quantities in (1.1) and (1.2) are dimensionless and related to the dimensional quantities in /3/ by the formula

$$a\{w, z, \alpha, \beta, -\rho, s, \varkappa^{-1}\} = \{W, S, \alpha_1, \alpha_2, n, \tau, \varkappa_0^{-1}\}, \varepsilon^2 = h(a\gamma)^{-1}, \quad \Psi = EFa^2\varepsilon h, \quad X = E\gamma\varepsilon^4 q, \quad \gamma^2 = 12(1-\nu^2)$$

Here a is the characteristic dimension of the domain D, z is the middle surface of an ideal shell, $z(s) = 0$ for $s \in \Gamma$, $\xi\zeta(\alpha, \beta)$ is the dimensionless initial deflection and $|\xi| \ll 1$. The

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* Prikl.Matem.Mekhan.,44,No.5,892-904,1980

functions $q(\alpha, \beta), z(\alpha, \beta)$ are considered sufficiently smooth. The boundary conditions correspond to: 1) moving hinge support; 2) sliding clamping of the edge; 3) absolutely rigid clamping of the edge.

Let $x^* \equiv (w^*, F^*)$ denote the fundamental solution of the problem (1.1), (1.2) for $\xi = 0$, and let us investigate the buckling of the appropriate equilibrium as the load changes. We assume the load to depend on a single parameter p and the buckling of the fundamental equilibrium of an ideal shell to appear as buckling at the bifurcation point p_0. Assuming

$$w = w^* + \omega, \quad F = F^* + \psi, \quad p = p_0 + \lambda \tag{1.3}$$

we obtain a system of equations in operator form

$$M_0 x = \Pi x + \sum_{m=1} \lambda^m C_m x - \xi T x + \xi \sum_{m=0} \lambda^m R_m, \quad M_0 x \equiv (\varepsilon^2 \Delta^2 \omega + [z, \psi] - [w^*(p_0), \psi] - [F^*(p_0), \omega], \tag{1.4}$$

$$-\varepsilon^2 \Delta^2 \psi + [z, \omega] - [w^*(p_0), \omega]), \quad x = (\omega, \psi), \ \Pi x = ([\omega, \psi], \ 1/2\,[\omega, \omega]), \quad Tx = ([\zeta, \psi], [\zeta, \omega])$$

$$-R_m = ([\zeta, F_m^*], [\zeta, w_m^*]), \quad C_m x = ([w_m^*, \psi] + [F_m^*, \omega], [w_m^*, \omega]); \quad \{w_m^*, F_m^*\} = \frac{1}{m!}\frac{\partial^m}{\partial p^m}\{w^*, F^*\}\Big|_{p=p_0}$$

together with boundary conditions of the form (1.2) from (1.1) and (1.2) for small perturbations ω, ψ, λ.

2. Application of the Liapunov–Schmidt operator method. As a result of linearizing the problems (1.1), (1.2) relative to x^*, we have a system

$$M_0 x = 0, \quad x = (\omega, \psi) \tag{2.1}$$

together with boundary conditions of the form (1.2).

Let a system of n vector-eigenfunctions correspond to the eigenvalue p_0 of the operator M_0. Orthonormalizing it relative to the metric of the space E^1 that is introduced in /9/, and letting $\varphi_i \equiv (\omega_i, \psi_i)$ denote the vector-eigenfunctions obtained, we have

$$\langle \varphi_i, \varphi_j \rangle_{E_1} = \delta_{ij}, \quad i, j = 1, 2, \ldots, n \tag{2.2}$$

where δ_{ij} is the Kronecker delta. Then because of the formal self-adjointness of the operator M_0, the Schmidt operator M_1 can be constructed in the form /5–7/

$$M_1 x = M_0 x + \sum_{i=1}^{n} a_i \mu_i \varphi_i, \quad \mu_i = \langle x, \varphi_i \rangle_{E_1}, \quad a_i \int_D (\omega_i^2 + \psi_i^2) AB\, d\alpha\, d\beta = 1, \quad i = 1, 2, \ldots, n \tag{2.3}$$

Now, by using (2.3) we obtain the equation

$$M_1 x = \Pi x + \sum_{m=1} \lambda^m C_m x - \xi T x + \xi \sum_{m=0} \lambda^m R_m + \sum_{i=1}^{n} a_i \mu_i \varphi_i \tag{2.4}$$

Seeking the solution x in the form of a series

$$x = \sum_{k+j+\delta \geqslant 1} x_{(k) j \delta} \mu_1^{k_1} \mu_2^{k_2} \ldots \mu_n^{k_n} \lambda^j \xi^\delta; \quad (k) = k_1, k_2, \ldots, k_n, \quad k = k_1 + k_2 + \ldots + k_n, \quad x_{(k)j\delta} = (\omega_{(k)j\delta}, \psi_{(k)j\delta}); \tag{2.5}$$

$$k, j, \delta \geqslant 0$$

where μ_i, λ, ξ are small numbers, we find equations to determine $x_{(k)j\delta}$ from (2.4)

$$M_1 x_{(1)00} = a_i \varphi_i \quad (k_i = 1, \ k_j = 0, \ i \neq j, \ i = 1, 2, \ldots, n) \tag{2.6}$$

$$M_1 x_{(0)10} = 0, \ M_1 x_{(0)01} = R_0, \ M_1 x_{(k)j\delta} = \sum_{\eta + i = j} C_i x_{(k)\eta\delta} + \Sigma' \Big([\omega_{(l)\eta r}, \psi_{(m)\gamma t}], \ \frac{1}{2}[\omega_{(l)\eta r}, \omega_{(m)\gamma t}]\Big) - ([\zeta, \psi_{(k)j\beta}], [\zeta, \omega_{(k)j\beta}]) -$$

$$\rho_0([\zeta, F_j^*], [\zeta, w_j^*]) \equiv f_{(k)j\delta}, \ k + j + \delta \geqslant 2, \ \beta = \delta - 1, \ (m) = m_1, m_2, \ldots, m_n; \ (l) = l_1, l_2, \ldots, l_n$$

Here $\rho_0 = 1$ if $(k) = 0, \delta = 1, j \geqslant 1$, and $\rho_0 = 0$ in all other cases. The boundary conditions have the form (1.2). Summation over the symbol Σ' occurs over all subscripts $m_i + l_i = k_i$, $\eta + \gamma = j$, $r + t = \delta$, where $m_i, l_i, \eta, \gamma, r, t$ are nonnegative integers. According to the generalized Schmidt lemma /5/, there exists a reciprocal linear bounded operator M_1^{-1}. Hence, all the problems (2.6), (2.7) are solvable. In particular, we have $x_{(1)00} = \varphi_i$ for $k_i = 1, k_j = 0, (i \neq j)$ and $x_{(0)10} = 0$. By using (2.6) we obtain a system of bifurcation equations /6/ from the expressions for μ_i in (2.3)

$$\Phi_i \equiv a_0^{(i)} \xi + \lambda c_m^{(i)} \mu_m + a_{ml}^{(i)} \mu_m \mu_l + b_{mlk}^{(i)} \mu_m \mu_l \mu_k + \ldots = 0 \tag{2.7}$$

where i varies between 1 and n and the ordinary summation rule is used. Applying (25.6) in /5/, we find the formula (6) from /9/ to determine the coefficients in (2.3) from (2.5), (2.6) but with ψ_0^*, ψ_1^* replaced, respectively, by F_0^*, F_1^*.

For $i = j$ conditions (2.2) permit evaluation of the constant factors $e_i \neq 0$ to the accuracy to which the vector eigenfunctions $\varphi_i = e_i X_i$ are determined, where X_i also satisfy the system (2.1). By using the change of variables

$$\nu_i = \mu_i e_i, \quad \varphi_i = e_i X_i, \quad a_0^{(i)} = e_i D^{(i)}, \quad c_m^{(i)} = e_i e_m C_m^{(i)}, \quad a_{ml}^{(i)} = e_i e_m e_l A_{ml}^{(i)}, \quad b_{mlk}^{(i)} = e_i e_m e_l e_k B_{mlk}^{(i)}$$

it can be seen that the system of bifurcation equations (2.7) and the asymptotic representations (2.5) result in a form independent of the amplitudes e_i. Therefore, arbitrary numbers not equal to zero can be taken as e_i in the calculations. In particular, it can be assumed $e_i = 1$ or $e_i = [\max_D |X_i|]^{-1}$, $i = 1, 2, \ldots, n$. This simple fact permits elimination of the uncertainty in the Koiter theory for a linear combination of buckled modes in the initial post-critical stage /17/.

Therefore, the problem of constructing equilibrium modes adjacent to x^* in the neighborhood of p_0 is reduced to the problem of seeking all real solutions of the system (2.7). After the solutions of this system have been found, the asymptotic representations of the equilibrium are obtained from (2.5). In the case of an imperfect shell $(\xi \neq 0)$, surfaces of values of the critical loads $p^*(A_1, A_2, \ldots, A_n)$ are formed in a sufficiently small neighborhood of the bifurcation point p_0, where $A_i = a_0^{(i)}$. To determine them, we shall solve the system (2.7) in combination with the necessary condition for buckling of the fundamental equilibrium /18/

$$\Phi_{n+1} \equiv \det \| \partial\Phi_i / \partial\mu_j \| = 0 \quad (i, j = 1, 2, \ldots, n) \tag{2.8}$$

Such a method permits finding the surface of critical load values as a function of n parameters characterizing the shell imperfections without seeking all the branches of the solutions of the system (2.7).

Let A_i be functions of a certain parameter s and let the vector $X_0 = \{\mu_1^0, \mu_2^0, \ldots, \mu_n^0, \lambda\}$ be the solution of the system (2.7), (2.8) for some value $s = s_0$. Differentiating its equation with respect to s, we obtain the system

$$\sum_{j=1}^{n} \frac{\partial\Phi_j}{\partial\mu_j}\frac{\partial\mu_j}{\partial s} + \frac{\partial\Phi_i}{\partial\lambda}\frac{\partial\lambda}{\partial s} = 0 \quad (i = 1, 2, \ldots, n+1) \tag{2.9}$$

For $s_1 = s_0 + \Delta s$ the solution of (2.7), (2.8) is found by using the Newton iterations

$$X_{k+1} = X_k - D^{-1}(X_k, s_1)\,\Phi(X_k, s_1), \quad X_k = \{\mu_1^{(k)}, \mu_2^{(k)}, \ldots, \mu_n^{(k)}, \lambda\}, \quad \Phi = \{\Phi_1, \Phi_2, \ldots, \Phi_{n+1}\} \tag{2.10}$$

Here D^{-1} is the inverse matrix to the matrix of the system (2.9), and $k = 0, 1, \ldots, l$, where l is the given number of iterations. Formulas (2.10) permit construction of the critical values $p^*(s)$ along a certain given path governed by the law of variation of A_i on s. Along this path the value p^* is a limit point, with the exception of the case when the rank of the whole matrix is less than $n+1$ and p^* $(\neq p_0)$ is a point of secondary bifurcation. Convergence of the iterations (2.10) drops as it is approached, since $\det(D)$ tends to zero. Let us note that the points of second bifurcation arouse special interest /2/ since they are characterized by an abrupt qualitative change in the system behavior.

3. Nonaxisymmetric buckling of shells of revolution in many eigenmodes.

The solution, based on the Koiter theory, is represented in /19/ for the problem of the initial stage in nonaxisymmetric buckling in one eigenmode for a rigidly clamped spherical dome subjected to a load distributed uniformly over a circular domain with center at the apex. Computational formulas are obtained in this section for the analysis of the initial stage of nonaxisymmetric buckling in many eigenmodes of the axisymmetric equilibrium x^* of arbitrary shells of revolution closed at the apex.

We derive the equilibrium equation of a shell of revolution subjected to the axisymmetric load $q = q(r, p)$ from (1.1), (1.2) for $A = 1, B = \alpha = r, \beta = \theta$. The boundary value problems obtained for $\xi = 0$ and any p have the axisymmetric solutions $x^*(r) = (w^*(r), F^*(r))$ that is determined from the system of nonlinear equations with the boundary conditions

$$\varepsilon^2 Au + uv - \theta_* v + \varphi(r) = 0, \quad \varepsilon^2 Av - \frac{1}{2}u^2 + \theta_* u = 0 \tag{3.1}$$

$$A(\;) \equiv -r\frac{d}{dr}\frac{1}{r}\frac{d}{dr}r(\;), \quad \varphi(r) = \varphi(r, p) = \int_0^r q(\tau, p)\,\tau\,d\tau, \quad u(r, p) = \frac{\partial w^*}{\partial r}, \quad v(r, p) = \frac{\partial F^*}{\partial r}, \quad \frac{\partial z}{\partial r} = \theta_*(r)$$

$$1)\ \left[\frac{du}{dr} + \nu u\right]_{r=1} = v(1) = 0, \quad 2)\ u(1) = v(1) = 0, \quad 3)\ u(1) = \left[\frac{dv}{dr} - \frac{\nu}{r}v\right]_{r=1} = 0, \quad \left|\frac{u}{r}, \frac{v}{r}\right|_{r=0} < \infty$$

Let us assume that the buckling of the subcritical equilibrium of an ideal shell of revolution is manifest at the bifurcation point p_0 in terms of the nonaxisymmetric buckling. Following /20/, we shall seek the solution of the boundary value problems (2.1), (1.2) for $A = 1,\ B = \alpha = r,\ \beta = \theta$ in the form

$$(\omega, \psi) = \cos n\theta x_n(r), \quad x_n(r) = (w_n, f_n) \tag{3.2}$$

Here n is an integer. After separating variables, we obtain a system with boundary conditions /20/

$$l_n^{(1)}x_n \equiv \varepsilon^2 L_n^2 w_n + \frac{1}{r}(\theta_* - u_0)'\left(f_n' - \frac{n^2}{r}f_n\right) + \frac{1}{r}(\theta_* - u_0) f_n'' - \frac{1}{r} v_0'\left(w_n' - \frac{n^2}{r}w_n\right) - \frac{1}{r} v_0 w_n'' = 0 \tag{3.3}$$

$$l_n^{(2)}x_n \equiv -\varepsilon^2 L_n^2 f_n + \frac{1}{r}(\theta_* - u_0)'\left(w_n' - \frac{n^2}{r}w_n\right) + \frac{1}{r}(\theta_* - u_0) w_n'' = 0$$

$$L_n(\ ) = (\ )'' + \frac{1}{r}(\ )' - \frac{n^2}{r^2}(\ ), \quad f_n = O(r^2), \quad w_n = O(r^2) \quad u(p_0) = u_0, \quad v(p_0) = v_0, \quad (\ )' = \frac{\partial(\ )}{\partial r}$$

$$1)\ w_n = w_n'' + \nu w_n' = f_n = f_n' = 0, \quad 2)\ w_n = w_n' = f_n = f_n' = 0$$

$$3)\ w_n = w_n' = f_n'' - \nu f_n' + \nu n^2 f_n = 0, \quad f_n''' - (2 + \nu) n^2 f_n' + 3n^2 f_n + (\nu - 1) f_n' = 0$$

to determine $w_n(r), f_n(r)$.

Exactly the same problems are obtained if the solution is sought in the form (3.2) but with $\cos n\theta$ replaced by $\sin n\theta$.

Let two vector eigenfunctions x_s, x_m correspond to the eigenvalue p_0 of the problems (3.3). Then four vector eigenfunctions correspond to the same value of the initial linearized problem, and we write them in the form

$$\varphi_1 = \cos s\theta\,(\gamma_1, \delta_1), \quad \varphi_2 = \cos m\theta\,(\gamma_2, \delta_2), \quad \varphi_3 = \sin m\theta\,(\gamma_1, \delta_2)$$

$$\varphi_4 = \sin s\,\theta\,(\gamma_1, \delta_1), \quad \gamma_i = \gamma_i(r), \quad \delta_i = \delta_i(r), \quad i = 1, 2$$

Here s, m are integers, where $s \neq {}^1/_2\, m,\ s < m$. It can be shown that only two of the vector eigenfunctions φ_i are linearly independent. Hence, we shall later consider the case of interaction between two modes φ_1 and φ_2 containing only cosines. For $k = 2,\ k_1 = k_2 = 1,\ j = \delta = 0$, we have from (2.6) for $x_{1100} = (\omega_{1100}, \psi_{1100})$

$$M_1 x_{1100} = ([\omega_1, \psi_2] + [\omega_2, \psi_1], [\omega_1, \omega_2]) = f_{1100} \tag{3.4}$$

Hence, by applying φ_1 and φ_2 and seeking the solution x_{1100} in the form

$$x_{1100} = E(r)\cos(m - s)\,\theta + F(r)\cos(m + s)\,\theta \tag{3.5}$$

we obtain a system to determine $E = (E_1, E_2)$

$$l_{m-s}^{(1)} E = {}^1/_2\,(I_1 - I_2), \quad l_{m-s}^{(2)} E = {}^1/_2\,(I_3 - I_4) \tag{3.6}$$

with the boundary conditions in (3.3), but with the subscript n replaced by $m - s$ and x_n by E, and the system

$$l_{m+s}^{(1)} F = {}^1/_2\,(I_1 + I_2), \quad l_{m+s}^{(2)} F = {}^1/_2\,(I_3 + I_4) \tag{3.7}$$

to determine $F = (F_1, F_2)$, with the boundary conditions in (3.3) but with x_n replaced by F and the subscript n by $m + s$. Here

$$rI_1 = [\gamma_1, \delta_2, m] + [\delta_2, \gamma_1, s] + [\delta_1, \gamma_2, m] + [\gamma_2, \delta_1, s], \quad r^2 I_2 = 2sm\,([\delta_2][\gamma_1] + [\delta_1][\gamma_2]) \tag{3.8}$$

$$rI_3 = [\gamma_1, \gamma_2, m] + [\gamma_2, \gamma_1, s], \quad r^2 I_4 = 2\,sm\,[\gamma_1][\gamma_2], \quad [\gamma_1, \delta_2, m] = \gamma_1''(\delta_2' - m^2\delta_2 / r), \quad [\gamma_1] = \gamma_1' - \gamma_1 / r$$

For $m - s = 1$, by using the change of variable

$$x = rE_1' - E_1, \quad y = rE_2' - E_2, \quad |\, x/r,\ y/r\,|_{r=0} < \infty \tag{3.9}$$

the system (3.6) is converted into the boundary value problem

$$\varepsilon^2(rx'' - x' - 3\,x/r) + (\theta_* - u_0)\,y - v_0 x = V_1(r), \quad -\varepsilon^2(ry'' - y' - 3\,y/r) + (\theta_* - u_0)\,x = V_2(r) \tag{3.10}$$

$$2\,V_1 = r\,(\gamma_1'\delta_2' + \delta_1'\gamma_2') + 2\,s\,(m\,/\,r)\,(\gamma_1\delta_2 + \delta_1\gamma_2) - m^2\gamma_1'\delta_2 - m^2\,\delta_1'\gamma_2 - s^2\,\gamma_1\delta_2' - s^2\,\delta_1\gamma_2',\quad 2\,V_2 =$$

$$r\gamma_1'\gamma_2' - m^2\,\gamma_1'\gamma_2 - s^2\,\gamma_1\gamma_2' + 2\,sm\,\gamma_1\gamma_2/r,\quad 1)\ x'\,(1) + \nu x\,(1) = y\,(1) = 0$$

$$2)\ x\,(1) = y\,(1) = 0,\quad 3)\ x\,(1) = y'\,(1) - \nu y\,(1) = 0$$

Furthermore, from (2.6) we have

$$M_1 x_{2000} = ([\omega_1, \psi_1],\ {}^1/_2\,[\omega_1, \omega_1]) \tag{3.11}$$

for $k = k_1 = 2,\quad k_2 = 0,\ j = \delta = 0$

An equation of such type is obtained in the investigation of buckling in one eigenmode. Following /19/, we seek the solution (3.11) in the form

$$x_{2000} = \int_1^r \sigma_1(r)\,dr + B(r)\cos 2s\theta \tag{3.12}$$

Here $\sigma_1 = (-\beta_1, \alpha_1)$ is determined from the problem

$$\varepsilon^2\,[(r\,\beta_1')' - \beta_1\,/\,r] - v_0\,\beta_1 + (u_0 - \theta_*)\,\alpha_1 = g_1\,(r),\quad \varepsilon^2\,[(r\,\alpha_1')' - \alpha_1\,/\,r] - u_0\,\beta_1 + \theta_*\beta_1 = g_2\,(r) \tag{3.13}$$

$$g_1\,(r) = {}^1/_2\,[s^2\,(\gamma_1\delta_1/r)' - \gamma_1'\delta_1'],\quad g_2\,(r) = {}^1/_4\,[s^2\,(\gamma_1^2/r)' - \gamma_1'^2],\quad |\,\alpha_1/r,\ \beta_1/r\,|_{r=0} < \infty$$

$$1)\ \beta_1'\,(1) + \nu\beta_1\,(1) = \alpha_1\,(1) = 0,\quad 2)\ \beta_1\,(1) = \alpha_1\,(1) = 0,\quad 3)\ \beta_1\,(1) = \alpha_1'\,(1) - \nu\alpha_1\,(1) = 0$$

and the vector function $B = (B_1, B_2)$ from the system

$$l_{2s}{}^{(1)}B = h_1\,(r),\quad l_{2s}{}^{(2)}B = h_2\,(r),\quad h_1 = [\gamma_1, \delta_1, s] + [\delta_1, \gamma_1, s] + 2\,s^2\,[\delta_1]\,[\gamma_1],\quad h_2 = [\gamma_1, \gamma_1, s] + s^2\,[\gamma_1, \gamma_1] \tag{3.14}$$

with the boundary conditions in (3.3) but with the subscript n replaced by $2s$ and x_n by B.

The vector function x_{0200} is contructed in the form (3.12) but with s replaced by m, σ_1 by $\sigma_2 \equiv (-\beta_2, \alpha_2)$, and B by $D \equiv (D_1, D_2)$, which are determined, respectively, from the problems (3.3)—(3.14) but with changing there s_1, γ_1, δ_1, respectively, by m, γ_2, δ_2.

The coefficients of the system of bifurcation equations are derived from formula (6) in /9/ for $n=2$. Omitting the tedious calculations, we present the final formulas to evaluate these coefficients

$$c_i^{(i)} = 4\pi\int_0^1 \beta_i\,\frac{\partial\varphi}{\partial p}\,dr,\quad c_2^{(1)} = c_1^{(2)} = 0,\quad a_{ml}^{(i)} = 0,\quad b_{222}^{(1)} = b_{112}^{(1)} = b_{111}^{(2)} = b_{122}^{(2)} = 0\quad (i, m, l = 1, 2) \tag{3.15}$$

$$b_{111}^{(1)} = -4\pi\int_0^1 \Big\{g_1\beta_1 - \alpha_1 g_2 - \frac{1}{2}\,r\,(h_1B_1 + h_2B_2)\Big\}\,dr,\qquad b_{222}^{(2)} = -4\pi\int_0^1 \Big\{G_1\beta_2 - \alpha_2 G_2 - \frac{1}{2}\,r\,(H_1D_1 + H_2D_2)\Big\}\,dr$$

$$b_{122}^{(1)} = \frac{\pi}{2}\Big\{I - 8\int_0^1 (\beta_2 g_1 - \alpha_2 g_2)\,dr\Big\},\qquad b_{112}^{(2)} = \frac{\pi}{2}\Big\{I - 8\int_0^1 (\beta_1 G_1 - \alpha_1 G_2)\,dr\Big\},\qquad b_{122}^{(1)} = b_{112}^{(2)}$$

$$I = \int_0^1 \{E_1\,(I_1 - I_2) + E_2\,(I_3 - I_4) + F_1\,(I_1 + I_2) + F_2\,(I_3 + I_4)\}\,dr$$

where G_i, H_i are obtained from expressions for g_i, h_i, respectively, by using the replacement of s, γ_1, δ_1 by m, γ_2, δ_2.

To prove (3.15), there is used the identity

$$\int_0^1 \{[E_1, \delta_1, s] + [\delta_1, E_1, m - s] - 2\,(m - s)\,s\,[E_1]\,[\delta_1]\,r^{-1}\}\,\gamma_2 dr =$$

$$\{E_1'\delta_1'\gamma_2 - s^2E_1'\delta_1\gamma_2 + 2\,(m - s)\,sE_1\,(\delta_1'\gamma_2 - \delta_1\gamma_2 r^{-1}) - E_1\delta_1'\gamma_2' +$$

$$E_1\,(\delta_1\gamma_2 r^{-1})'\}_0^1 + \int_0^1 E_1\{[\gamma_2, \delta_1, s] + [\delta_1, \gamma_2, m] - 2ms\,[\gamma_2]\,[\delta_1]\,r^{-1}\}\,dr$$

which is confirmed by integration by parts. The equality $b_{122}^{(1)} = b_{112}^{(2)}$ is deduced from the two formulas preceding it in (3.15). This equality is the corollary of the conservativness of the construction. It is assumed in the derivation of (3.15) that $s \neq {}^1/_2\,m$ and $s \neq {}^1/_3\,m$ ($s < m$).

We now assume that n vector eigenfunctions correspond to the eigenvalue p_0 of the problem (3.3). Then we have n vector eigenfunctions of the form

$$\varphi_i = \cos m_i\,\theta\,(\gamma_i, \delta_i),\quad \gamma_i = \gamma_i\,(r),\quad \delta_i = \delta_i\,(r),\quad i = 1, 2, \ldots, n,\quad m_1 < m_2 < \ldots < m_n \tag{3.16}$$

for the eigenvalue p_0 of the initial linearized problem (2.1). Here $m_i \neq {}^1/_2\,m_j$, $m_i \neq {}^1/_3\,m_j$, ($j > i$).

The vector eigenfunctions $\sin m_i\theta(\gamma_i, \delta_i)$ are not included in the system (3.16) because of their linear dependence on φ_i. Determination of the coefficients of the series (2.5) is reduced to solving problems of either the form (3.4)—(3.10), or (3.11)—(3.14). Coefficients of the system of bifurcation equations are evaluated by formulas (3.15) with the appropriate change in subscript (m_i for s and m_j for m).

We consequently obtain that the system (2.7) reduces to the form

$$\Phi_i \equiv \mu_i\left(\sum_{k=1}^{n} a_{ik}\mu_k^2 + \lambda\right) + \xi d_i = 0 \quad (i=1, 2, \ldots, n), \quad d_i = A_i c_i^{-1}, \quad a_{ik} = h_{ik}c_i^{-1}, \quad c_i \neq 0, \quad h_{ik} = h_{ki} \qquad (3.17)$$

In the case the conditions $m_i \neq {}^1/_2\, m_j$ or $m_i \neq {}^1/_3\, m_j$ are disturbed, some of the zero coefficients in (3.15) will not be zero and the form of the bifurcation system (3.17) will be changed (see /2,17/, for instance). In order to find the solution of the system (3.17), we assume analogously to /9/

$$d_1 = \ldots\ldots = d_m = 0, \quad d_n \neq 0, \quad m = n-1 \qquad (3.18)$$

The first group of 2^m families of solutions of the system (3.17), (3.18), (2.8) is represented by formulas (9) in /9/. To find the second group of families of solutions, we assume

$$\mu_{k_1} = \mu_{k_2} = \ldots = \mu_{k_l} = 0, \quad 1 \leqslant k_j \leqslant m; \quad j = 1, 2, \ldots, l \qquad (3.19)$$

We equate the expression in parentheses in the first m equations of the system (3.17), (3.18) to zero. In the system of linear equations with respect to μ_i^2 obtained we extract the matrix that is obtained from the matrix of coefficients a_{ik} by cancelling columns with numbers $k_1, k_2, \ldots, k_l$ and rows with numbers $k_1, k_2, \ldots, k_{l-1}, n$. We denote the extracted matrix of order $n-l$ by $E^{k_1k_2\ldots k_l}$, and its determinant by E, i.e.,

$$E = \det(E^{k_1k_2\ldots k_l}) = \det\|a_{rs}\|, \quad 1 \leqslant r \leqslant m$$
$$r \neq k_1, k_2, \ldots, k_{l-1}, n; \quad 1 \leqslant s \leqslant n, \quad s \neq k_1, k_2, \ldots, k_{l-1}, \qquad (3.20)$$
$$k_l; \quad k_l \neq n$$

Solving the system of equations with the matrix $E^{k_1k_2\ldots k_l}$ we obtain for $i \neq k_j$ $(j = 1, 2, \ldots, l)$

$$\mu_i^2 = -\lambda E^{-1}H_i, \quad H_i = \sum_{r=1}^{m}{}' E_{ri} \qquad (3.21)$$

Here E_{ri} is the cofactor of the element a_{ri} of the matrix $E^{k_1k_2\ldots k_l}$, and the prime means that the summation is over subscripts which do not agree with $k_1, k_2, \ldots, k_{l-1}$. Substituting (3.21) into the last equation in (3.17), we obtain

$$p_0 - p = -\lambda = EH_n^{-1/_3}\left\{\xi d_n\left(E - \sum_{t=1}^{n}{}'' a_{nt}H_t\right)\right\}^{1/_3} \qquad (3.22)$$

Here the double prime means that the summation is over subscripts that do not agree with $k_1, k_2, \ldots, k_l$. All the families of solutions of this group are obtained by a change in the number l in (3.19) and by sampling l of the subscripts $k_1, k_2, \ldots, k_l$ from m elements. It can be shown that the number of solutions of the form (3.21), (3.22) equals $m\,2^{m-1}$. Let us note that the number of families of solutions here and in /9/ were computed relative to μ_i^2. Upon extracting the square root, different solutions can appear, which differ in sign in part or all of the μ_i. Each of these solutions generates its surface of critical loads although the values of λ are identical for them for $d_1 = \ldots = d_m = 0$, $d_n \neq 0$ (see Sect.4, below).

4. Spherical shell under uniform external pressure. A packet of numerical programs for the BESM-6 electronic computer was compiled by the formulas from Sect.3 by using finite differences in combination with matrix factorization and the procedure of continuation in the load parameter /19—21/. Let us present some results of computations for imperfect spherical shells.

Let q_0 be the classical value of the critical pressure for a complete sphere $\Lambda \equiv 2[3(1-\nu^2)]^{1/4}(H/h)^{1/2}$, $p_0 = p_H/q_0$, where H is the shell rise, h is its thickness, ν is the Poisson's ratio, and p_H is the critical pressure of the nonaxisymmetrical buckling of an ideal spherical shell. There then results from /20/ that two vector eigenfunctions with the harmonics $s = 11$, $m = 12$ and amplitudes determined from (3.1), (3.3) for $\theta_* = -r$ correspond to the least eigenvalue $p_0 = 0.790$ of the problem (2.1) in the case of rigid support of the edge for $\Lambda = 17$.

Evaluating the coefficients of the system of bifurcation equations by means of (3.15), we obtain that the critical loads $p^*(A_1, A_2)$ are determined as functions of the geometric imperfections from the system of equations

$$\frac{\partial V}{\partial \mu_k} = 0, \quad \det\left\|\frac{\partial^2 \eta}{\partial \mu_k \partial \mu_j}\right\| = 0, \quad k, j = 1, 2 \tag{4.1}$$

$$V = 910.74\mu_1^4 + 1023.58\mu_2^4 + 3871.80\mu_1^2\mu_2^2 + (p - p_0)(2123.50\mu_1^2 + 2377.44\mu_2^2) + \xi\,(A_1\mu_1 + A_2\mu_2)$$

Let us note that because of (3.15) the coefficients $b_{122}^{(1)}$ and $b_{122}^{(2)}$ are equal. However, to check the computation, they are evaluated independently by the formulas represented in (3.15). The discrepancy between the values obtained is 0.018%. Using (4.1), we find the coefficients

$$a_1 = a_{11} = 0.8578, \quad b_1 = a_{12} = 1.8233, \quad a_2 = a_{21} = 1.6286, \quad b_2 = a_{22} = 0.8636 \tag{4.2}$$

for the system (3.17) for $n = 2$. For an initial deflection $\xi\zeta_2(r)\cos 12\theta$ (in the $d_1 = 0$ plane), we find three critical values of shell snap-through governed by the formulas

$$p_2^{(i)} = p_0 - \eta_i\,(d_2\xi)^{2/3}, \quad |\xi| \ll 1, \quad i = 1, 2, 3, 4 \tag{4.3}$$

$$\eta_1 \equiv [1.5\,(3b_2)^{1/2}]^{2/3} \approx 1.798, \quad \eta_2 \equiv b_1(b_1 - b_2)^{-2/3} \approx 1.871, \quad \eta_3 = \eta_4 \equiv [1.5\Delta_2^{-1}\,(3\Delta_3\Delta_2^{-1})^{1/2}a_1]^{2/3} \approx 2.892$$

$$d_2 = \frac{-\pi}{c_2}\int_0^1 \left\{(\zeta_2'v_0)'\,\gamma_2 + (\zeta_2'u_0)'\,\delta_2 - \frac{m^2}{r}\,\zeta_2\,(v_0'\gamma_2 + u_0'\delta_2)\right\} dr, \quad \Delta_2 = a_1 - a_2, \quad \Delta_3 = a_1b_2 - a_2b_1$$

in a sufficiently small neighborhood of p_0, from formulas (9) in /9/ and (3.21), (3.22). We hence have four solutions

$$\mu_1^{(1)} = 0, \quad \mu_2^{(1)} = \left[\frac{1}{3}L_1b_2^{-1}\right]^{1/2}, \quad \mu_1^{(2)} = 0, \quad \mu_2^{(2)} = [L_2b_1^{-1}]^{1/2} \tag{4.4}$$

$$\mu_1^{(3)} = [(L_3 - b_1y^2)\,a_1^{-1}]^{1/2}, \quad y = \mu_2^{(3)} = -\,[{}^1/_3L_3\Delta_3\Delta_2^{-1}]^{1/2}$$

$$\mu_1^{(4)} = -\mu_1^{(3)}, \quad \mu_2^{(4)} = \mu_2^{(3)}, \quad L_i = -\lambda_i = p_0 - p_2^{(i)}, \quad i = 1, 2, 3, 4$$

For an initial deflection $\xi\zeta_1(r)\cos 11\theta$ (in the $d_2 = 0$ plane), we deduce

$$p_1^{(i)} = p_0 - \varkappa_i\,(d_1\xi)^{2/3}, \quad |\xi| \ll 1, \quad i = 5, 6, 7, 8, \quad \varkappa_5 = 1.795, \quad \varkappa_6 = 1.937, \quad \varkappa_7 = \varkappa_8 = 2.323 \tag{4.5}$$

from (9) in /9/ and (3.21), (3.22). Here d_1 is obtained from d_2 by replacing m by s and all the subscripts 2 by the subscripts 1. Formulas for solutions for $i = 5, 6, 7, 8$ are obtained, respectively, from (4.4) for $i = 1, 2, 3, 4$ by replacing a_1, b_1, a_2, b_2, respectively, by b_2, a_2, b_1, a_1, and moreover, $\mu_1^{(i)}, \mu_2^{(i)}$ by $\mu_2^{(i)}, \mu_1^{(i)}$. Values of $\mu_1^{(i)}, \mu_2^{(i)}$, and L_i for $i = 1$ and 5 correspond to buckling in one eigenmode /19/.

To construct the critical load surfaces as functionals of the geometric imperfections ξd_1 and ξd_2, we solve the system (3.17) numerically for $n = 2$, (3.18), (4.2) with the application of formulas (2.10). Let us introduce polar coordinates by setting $\xi d_1 = R\cos\alpha$, $\xi d_2 = R\sin\alpha$, where $0 \leqslant \alpha \leqslant 2\pi$. Continuing each of the solutions (4.3)—(4.5) along the angle α for fixed R we obtain that four surfaces $L_i(\xi d_1, \xi d_2)$ are located above the plane $L = 0$, where $i = 1, 2, 3, 4$. For $n = 2$ the system (3.17) possesses symmetry properties when the signs of d_i or μ_i are reversed. A section of these surfaces by a circular cylinder with radius $R = 0.01$ is represented in the Fig.1. The surface L_1 passes through the curve (4.3) for $i = 1$ and (4.5) for $i = 5$, which are obtained under the assumption of buckling in one eigenmode. The intersection of L_1 with the cylinder yields the curve 1 in the Fig.1, which takes the value $L \approx 0.0833$ at the points $\alpha = 0$, ${}^1/_2\pi$, π and the value $L \approx 0.0955$ at the points $\alpha = {}^1/_4\pi$, ${}^3/_4\pi$, close to the maximum values on theis curve. There are no singular points on the surface L_1, where $\det D = 0$. (The point $R = 0$ is not taken into account). The surfaces L_2, L_3, L_4 are interconnected along the rays α mentioned below to form a three-sheeted surface with self-intersections and reentries in the three-space $(\xi d_1, \xi d_2, L)$. We mention at once that the curves (4.3) for $i = 2$ and (4.5) for $i = 6$, that lie on these surfaces, consist of singular points at which $\det D = 0$. For $\alpha = 0$ the surface L_2 passes through the curve (4.5) for $i = 7$, for $\alpha = 0.5\pi$ through the curve of singular points (4.3) for $i = 2$, for $\alpha = \pi$ through the curve (4.5) for $i = 7$, and for $\alpha = 1.5\pi$ through the curve (4.3) for $i = 3$. Upon the traversal in α along this surface after a complete revolution, i.e., for $\alpha = 2\pi$ we will not return to

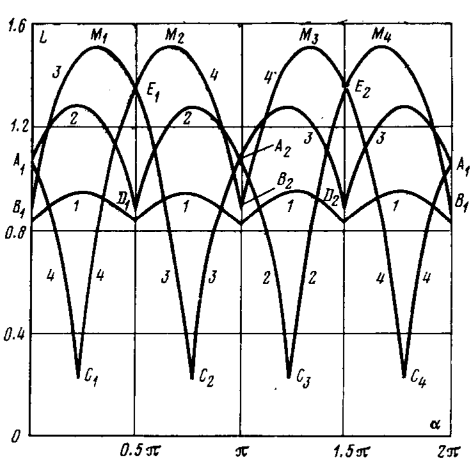

Fig.1

the curve (4.5) for $i=7$, but will arrive at the curve of singular points (4.5) for $i=6$, which lies below. Along this curve the surface L_2 goes over into the surface L_3 in the five-dimensional space of the variables $\xi d_1, \xi d_2, L, \mu_1, \mu_2$. Still another set of singular points is on the surface L_2 along the ray $\pi+\alpha_*$, where $\alpha_*=0.730$. The intersection of L_2 with a cylinder yields curve *2* in the Fig.1.

The surface L_3 passes through the curve of singular points (4.5) for $i=6$ for $\alpha=0$, through the curve (4.3) for $i=3$ for $\alpha=0.5\pi$, through the curve (4.5) for $i=8$ for $\alpha=\pi$, and through the curve (4.3) for $i=2$ for $\alpha=1.5\pi$. Upon a traversal in α along the surface L_3 after a complete rotation, we arrive at the curve (4.5) for $i=8$, along which the surface L_3 is connected to the surface L_4 in the five-dimensional space. Along the ray $\pi-\alpha_*$ there is a family of singular points on L_3. The intersection of L_3 with the cylinder yields the curve *3* in the Fig.1.

The surface L_4 passes through the curve (4.5) for $i=8$ for $\alpha=0$, through the curve (4.3) for $i=4$ for $\alpha=0.5\pi$, through the curve of singular points (4.5) for $i=6$ for $\alpha=\pi$, through the curve (4.3) for $i=4$ for $\alpha=1.5\pi$, and after a complete rotation in α arrives at the curve (4.5) for $i=7$. The surfaces L_2 and L_4 are connected along this last curve. Singular points are located on the surface L_4 along the rays α_* and $2\pi-\alpha_*$. The intersection of L_4 with the cylinder yields the curve *4* in the Fig.1.

Therefore, by making one complete rotation in α, we pass the surface L_2 and drop onto the surface L_3, which we traverse as a result of the second complete rotation in α, and we now drop onto the surface L_4. After the third complete rotation in α we pass the surface L_4 and return to the surface L_2. A further traversal duplicates the picture described above. Let us present the coordinates of the points noted in the figure. The points A_i, B_i, D_i, E_i have the ordinates $0.1073, 0.0899, 0.0868, 0.1342$, respectively. As has already been noted, the points B_i, D_i, C_j, where $i=1,2$ and $j=1,2,3,4$, are singular or secondary bifurcation points /2,22/. The points C_1, C_2, C_3, C_4 have the coordinates $(\alpha_*, y_*), (\pi-\alpha_*, y_*), (\pi+\alpha_*, y_*), (2\pi-\alpha_*, y_*)$, respectively, where $\alpha_*=0{,}730$, $y_*=0.0218$. The points $M_j(\alpha_j, 0.151)$ are of indubitable interest, where $\alpha_1=0.981$, $\alpha_2=\pi-\alpha_1$, $\alpha_3=\pi+\alpha_1$, $\alpha_4=2\pi-\alpha_1$. At these points curves *2, 3, 4* take on maximal values, which corresponds to the greatest reduction in the critical pressure for a given value of $R=|\xi|(d_1^2+d_2^2)^{1/2}=0.01$.

5. Conical shell under uniform external pressure.

There results from the numerical results in /23/ that nonaxisymmetrical buckling along two eigenmodes holds for a conical shell under uniform external pressure and rigid clamping of the edge when $\Lambda=24$ and $p_0=0.242$, where the appropriate harmonics have the number $s=10$ and $m=11$. In this case computations by the formulas of Sect.3 result in the potential function

$$V=42.134\mu_1^4+44.722\mu_2^4+170.04\mu_1^2\mu_2^2+(p-p_0)(1946.81\mu_1^2+2183.15\mu_2^2)+\xi(A_1\mu_1+A_2\mu_2) \qquad (5.1)$$

For the initial deflection $\xi\zeta_2(r)\cos 11\theta$ (in the $d_1=0$ plane) we obtain (4.3) and (4.4), where $\eta_1=0.6635$, $\eta_2=0.6767$, $\eta_3=\eta_4=1.1536$. For the initial deflection $\xi\zeta_1(r)\cos 10\theta$ (in the $d_2=0$ plane), we obtain (4.5), where $\varkappa_5=0.6515$, $\varkappa_6=0.7335$, $\varkappa_7=\varkappa_8=0.8299$. The location of the surfaces L_i is analogous to the case of the spherical shell described above; $C_1(0.7127, 0.0086)$; $M_1(0.982, 0.060)$.

6. Spherical shell under radially varying pressure.

Let us consider the nonaxisymmetric buckling of a rigidly clamped spherical shell under a pressure distributed according to the law $q=4p\sin(\pi r/2)$. The case of buckling in two eigenmodes holds for $\Lambda=40$, $p_0=0.743$, where $s=32$ and $m=33$. Computations by the formulas of Sect.3 result in the system (4.1), where the function V now has the form

$$V=1884.73\mu_1^4+1969.78\mu_2^4+7708.95\mu_1^2\mu_2^2+(p-p_0)(5905.88\mu_1^2+6211.91\mu_2^2)+\xi(A_1\mu_1+A_2\mu_2) \qquad (6.1)$$

For the initial deflection $\xi\zeta_2(r)\cos 33\theta$ (in the $d_1=0$ plane) we obtain (4.3), (4.4), where $\eta_1=1.6237$, $\eta_2=1.7029$, $\eta_3=\eta_4=2.4788$. For the initial deflection $\xi\zeta_1(r)\cos 32\theta$ (in the $d_2=0$ plane) we obtain (4.5), where $\varkappa_5=1.6272$, $\varkappa_6=1.7390$, $\varkappa_7=\varkappa_8=2.2185$. The disposition of the surfaces L_i is analogous to the preceding; $C_1(0.759, 0.018)$; $M_1(0.982, 0.119)$.

For a rigidly clamped spherical shell subjected to a pressure distributed according to the law $q=4pr^2$ the case of buckling holds, for instance, for $\Lambda=40$, $p_0=0.778$, where $s=33, m=34$. Analogously, we find

$$V=1344.68\mu_1^4+1397.08\mu_2^4+5486.65\mu_1^2\mu_2^2+(p-p_0)(5652.02\mu_1^2+5929.83\mu_2^2)+\xi(A_1\mu_1+A_2\mu_2)$$

For the initial deflection $\xi\zeta_2(r)\cos 34\theta$ we obtain (4.3), (4.4), where $\eta_1=1.4706$, $\eta_2=1.5419$, $\eta_3=2.24728$, $\eta_3=\eta_4$. For the initial deflection $\xi\zeta_1(r)\cos 33\theta$ we obtain (4.5), where $\varkappa_5=1.4754$, $\varkappa_6=1.5770$, $\varkappa_7=\varkappa_8=2.0085$. The arrangement of the surfaces L_i is the same; $C_1(0.759, 0.020)$; $M_1(0.982, 0.131)$.

The author is grateful to I. I. Vorovich, V. I. Iudovich, and P. E. Tovstik for discussing the results.

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Translated by M.D.F.